

The spreading fronts of an infective environment in a man-environment-man epidemic model*

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Abstract. A reaction-diffusion model is investigated to understand infective environments in a man-environment-man epidemic model. The free boundary is introduced to describe the expanding front of an infective environment induced by fecally-orally transmitted disease. The basic reproduction number $R_0^F(t)$ for the free boundary problem is introduced, and the behavior of positive solutions to the reaction-diffusion system is discussed. Sufficient conditions for the bacteria to vanish or spread are given. We show that, if $R_0 \leq 1$, the bacteria always vanish, and if $R_0^F(t_0) \geq 1$ for some $t_0 \geq 0$, the bacteria must spread, while if $R_0^F(0) < 1 < R_0$, the spreading or vanishing of the bacteria depends on the initial number of bacteria, the length of the initial habitat, the diffusion rate, and other factors. Moreover, some sharp criteria are given.

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1 Introduction

Recently, many mathematical models have been proposed to investigate the spatial spread of infectious diseases epidemics (see [1, 2, 3, 28]). To understand

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the dynamics of fecally-orally transmitted diseases in the European Mediterranean regions, Capasso and Maddalena [4] have proposed an epidemic reaction-diffusion model described by the following coupled parabolic system:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) - a_{11}u(x,t) + a_{12}v(x,t), & (x,t) \in \Omega \times (0, +\infty), \\ \frac{\partial v(x,t)}{\partial t} = -a_{22}v(x,t) + G(u(x,t)), & (x,t) \in \Omega \times (0, +\infty), \\ \frac{\partial u}{\partial \eta} + \alpha u = 0, & (x,t) \in \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \overline{\Omega}, \end{cases} \quad (1.1)$$

where $u(x,t)$ and $v(x,t)$ represent the spatial densities of bacteria and the infective human population, respectively, at a point x in the habitat $\Omega \in \mathbb{R}^n$, and at time $t \geq 0$, and $\partial/\partial\eta$ denotes the outward normal derivative. The positive constant d denotes the diffusion constant of the bacteria, $1/a_{11} > 0$ is the mean lifetime of the bacteria in the environment, the term $-a_{11}u$ denotes the natural growth rate of the bacterial population, $1/a_{22} > 0$ is the mean infectious period of an infective human, the term $-a_{22}v$ describes the natural damping of the infective population due to the finite mean duration of the infectiousness of humans, $a_{12} > 0$ is the multiplicative factor of the infectious bacteria due to the human population, and the term $a_{12}v$ is the contribution of the infective humans to the growth rate of the bacteria. The last term $G(u)$ is the infection rate of the humans under the assumption that the total susceptible human population is constant during the evolution of the epidemic. This kind of mechanism is used to interpret other epidemics with oro-faecal transmission such as typhoid fever, infectious hepatitis, polyomelitis, and the like ; see [4, 5] and the references therein for more details.

Assume that

$$(A1) \quad G \in C^1([0, \infty)), G(0) = 0, G'(z) > 0, \forall z \geq 0;$$

$$(A2) \quad \frac{G(z)}{z} \text{ is decreasing and } \lim_{z \rightarrow +\infty} \frac{G(z)}{z} < \frac{a_{11}a_{22}}{a_{12}}.$$

An example is $G(z) = \frac{a_{21}z}{1+z}$ with $a_{21} > 0$.

For the corresponding O.D.E. system of (1.1),

$$\begin{cases} \frac{du(t)}{dt} = -a_{11}u(t) + a_{12}v(t), & t > 0, \\ \frac{dv(t)}{dt} = -a_{22}v(t) + G(u(t)), & t > 0, \end{cases} \quad (1.2)$$

linearization and spectrum analysis show that a threshold parameter $R_0(= \frac{G'(0)a_{12}}{a_{11}a_{22}})$ exists such that if $0 < R_0 < 1$, then the epidemic always tends to extinction, while for $R_0 > 1$, a nontrivial endemic level appears which is globally asymptotically stable in the positive quadrant.

For problem (1.1), in which the bacteria diffuse but the infective human population does not, the authors in [4] introduced a threshold parameter R_0^D ($:= \frac{G'(0)a_{12}}{(a_{11}+d\lambda_1)a_{22}}$) such that for $0 < R_0^D < 1$, the epidemic eventually tends to extinction, while for $R_0^D > 1$ a globally asymptotically stable spatially inhomogeneous stationary endemic state appears, where λ_1 is the first eigenvalue of the boundary value problem

$$-\Delta\phi = \lambda\phi \text{ in } \Omega \text{ with } \frac{\partial\phi}{\partial\eta} + \alpha\phi = 0 \text{ on } \partial\Omega.$$

To understand the whole dynamical structure of solutions to (1.1) and its corresponding reaction systems, traveling waves and entire solutions were widely studied. The existence, uniqueness and stability of traveling waves were established in [21, 23, 24, 25, 26, 27]. Recently, Wu [25] considered entire solutions of a bistable reaction-diffusion system (1.1) in the bistable case, and proved the existence of entire solutions that behave like two monotone increasing traveling wave solutions propagating from both sides of the x -axis. The time-delayed and diffusive model has been considered in [23] and entire solutions have been given. It was shown that there exist a great diversity of different types of entire solutions of reaction-diffusion equations, which are different from traveling wave solutions.

It must be pointed out that the solution of (1.1) in a fixed (bounded or unbounded) domain is always positive for any $t > 0$ no matter what the non-negative nontrivial initial data are. This means that bacteria spread and the whole environment is infected immediately even though the infection is limited to a small part of population at the beginning. This does not match the reality that bacteria always spread gradually. The traveling wave solutions and entire solutions play a key role in developing a full understanding of the transient dynamics and the structure of the global attractor, but none of those solutions can explain the gradual expanding process.

To describe such a gradual spreading process and changing of the domain considered, the free boundary has been introduced in many applied areas, especially the well-known Stefan condition used to describe the spreading process at the boundary. The Stefan condition was used originally to describe the melting of ice in contact with water [19]; it was then used in modeling oxygen in the muscle [7], wound healing [6], and more recently the spreading of species in ecological models [8, 9, 10, 11, 13, 15, 17, 18, 22].

For emerging and re-emerging infectious bacteria, the expanding of bacteria usually starts at a source location and spreads over areas where contact transmission occurs. It is crucial and interesting to study how bacteria spread spatially to a larger area to cause an environmental problem. We will focus on

the changing of the infected habitat and consider an epidemic model with the free boundary, which describes the spreading front of bacteria. For simplicity, assume that the human population in the whole habitat $(-\infty, \infty)$ is constant, and that the environment in $g(t) < x < h(t)$ is infected by bacteria, the density of which is denoted by $u(x, t)$ with the infective human population denoted by $v(x, t)$, and no bacteria or infective humans in the remaining portion of the environment. The right spreading front of the infected environment is represented by the free boundary $x = h(t)$. Assuming that $h(t)$ grows at a rate proportional to the bacteria population gradient at the front [17], the conditions on the right front (free boundary) are

$$u(h(t), t) = 0, \quad -\mu \frac{\partial u}{\partial x}(h(t), t) = h'(t).$$

Similarly, the conditions on the left front (free boundary) are

$$u(g(t), t) = 0, \quad -\mu \frac{\partial u}{\partial x}(g(t), t) = g'(t).$$

In such a case, we have the problem for $u(x, t)$ and $v(x, t)$ with free boundaries $x = g(t)$ and $x = h(t)$ such that

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = d \frac{\partial^2 u(x, t)}{\partial x^2} - a_{11}u(x, t) + a_{12}v(x, t), & g(t) < x < h(t), t > 0, \\ \frac{\partial v(x, t)}{\partial t} = -a_{22}v(x, t) + G(u(x, t)), & g(t) < x < h(t), t > 0, \\ u(x, t) = 0, & x = g(t) \text{ or } x = h(t), t > 0, \\ g(0) = -h_0, g'(t) = -\mu \frac{\partial u}{\partial x}(g(t), t), & t > 0, \\ h(0) = h_0, h'(t) = -\mu \frac{\partial u}{\partial x}(h(t), t), & t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & -h_0 \leq x \leq h_0, \end{cases} \quad (1.3)$$

where $x = g(t)$ and $x = h(t)$ are the moving left and right boundaries to be determined, h_0 and μ are positive constants, and the initial functions u_0 and v_0 are nonnegative and satisfy

$$\begin{cases} u_0 \in C^2([-h_0, h_0]), u_0(\pm h_0) = 0 \text{ and } 0 < u_0(x), & x \in (-h_0, h_0), \\ v_0 \in C^2([-h_0, h_0]), v_0(\pm h_0) = 0 \text{ and } 0 < v_0(x), & x \in (-h_0, h_0). \end{cases} \quad (1.4)$$

The remainder of this paper is organized as follows. In the next section, the global existence and uniqueness of the solution to (1.3) are proved using a contraction mapping theorem, and a comparison principle is presented. Section 3 is devoted to sufficient conditions for the bacteria to vanish. Section 4 deals with the case and conditions for the bacteria to expand and the whole environment become infected. Finally, we give a brief discussion in Section 5.

2 Existence and uniqueness

In this section, we first present the following local existence and uniqueness result using the contraction mapping theorem and then show global existence using suitable estimates.

Theorem 2.1 *For any given (u_0, v_0) satisfying (1.4), and any $\alpha \in (0, 1)$, there is a $T > 0$ such that problem (1.3) admits a unique solution*

$$(u, v; g, h) \in [C^{1+\alpha, (1+\alpha)/2}(D_T)]^2 \times [C^{1+\alpha/2}([0, T])]^2;$$

moreover,

$$\|u\|_{C^{1+\alpha, (1+\alpha)/2}(D_T)} + \|v\|_{C^{1+\alpha, (1+\alpha)/2}(D_T)} + \|g\|_{C^{1+\alpha/2}([0, T])} + \|h\|_{C^{1+\alpha/2}([0, T])} \leq C \quad (2.1)$$

where $D_T = \{(x, t) \in \mathbb{R}^2 : x \in [g(t), h(t)], t \in [0, T]\}$, C and T depend only on $h_0, \alpha, \|u_0\|_{C^2([-h_0, h_0])}$ and $\|v_0\|_{C^2([-h_0, h_0])}$.

Proof: As in [29], we first straighten the double free boundary fronts by making the following change of variable:

$$y = \frac{2h_0x}{h(t) - g(t)} - \frac{h_0(h(t) + g(t))}{h(t) - g(t)}, \quad w(y, t) = u(x, t), \quad z(y, t) = v(x, t).$$

Then (1.3) can be transformed into

$$\begin{cases} w_t = Aw_y + Bw_{yy} - a_{11}w(y, t) + a_{12}z(y, t), & t > 0, \quad -h_0 < y < h_0, \\ z_t = Az_y - a_{22}z(y, t) + G(w(y, t)), & t > 0, \quad -h_0 < y < h_0, \\ w = 0, \quad h'(t) = -\frac{2h_0\mu}{h(t)-g(t)} \frac{\partial w}{\partial y}, & t > 0, \quad y = h_0, \\ w = 0, \quad g'(t) = -\frac{2h_0\mu}{h(t)-g(t)} \frac{\partial w}{\partial y}, & t > 0, \quad y = -h_0, \\ h(0) = h_0, \quad g(0) = -h_0, \\ w(y, 0) = w_0(y) := u_0(y), \quad z(y, 0) = z_0(y) := v_0(y), \quad -h_0 \leq y \leq h_0, \end{cases} \quad (2.2)$$

where $A = A(h, g, y) = y \frac{h'(t)-g'(t)}{h(t)-g(t)} + h_0 \frac{h'(t)+g'(t)}{h(t)-g(t)}$, and $B = B(h, g) = \frac{4h_0^2d}{(h(t)-g(t))^2}$. This transformation changes the free boundaries $x = h(t)$ and $x = g(t)$ to the fixed lines $y = h_0$ and $y = -h_0$ respectively; therefore, the equations become more complex, because now the coefficients in the first and second equations of (2.2) contain unknown functions $h(t)$ and $g(t)$.

The rest of the proof uses by the contraction mapping argument as in [10, 29] with suitable modifications; we omit it here. \square

To show the global existence of the solution, we need the following estimate.

Lemma 2.2 *Let $(u, v; g, h)$ be a solution to (1.3) defined for $t \in (0, T_0]$ for some $T_0 \in (0, +\infty)$. Then there exist constants C_1 and C_2 independent of T_0 such that*

$$\begin{aligned} 0 < u(x, t) &\leq C_1 \text{ for } g(t) < x < h(t), \ t \in (0, T_0], \\ 0 < v(x, t) &\leq C_2 \text{ for } g(t) < x < h(t), \ t \in (0, T_0]. \end{aligned}$$

Proof: The positivity of u and v are obvious, since the initial values are non-trivial and nonnegative and the system is quasi-increasing. Now let us consider its upper bounds. Note that $\lim_{z \rightarrow +\infty} \frac{G(z)}{z} < \frac{a_{11}a_{22}}{a_{12}}$ by the assumption (A2); therefore, there exist C_1 and C_2 such that

$$\begin{aligned} C_1 &\geq u_0(x), \ C_2 \geq v_0(x) \text{ in } [-h_0, h_0], \\ -a_{11}C_1 + a_{12}C_2 &< 0, \ -a_{22}C_2 + G(C_1) < 0. \end{aligned}$$

Define

$$\begin{aligned} M_1 &= M_1(u, T_0) = \max_{[g(t), h(t)] \times [0, T_0]} u(x, t), \\ M_2 &= M_2(G, C_1, T_0) = \max_{0 \leq z \leq \max\{C_1, M_1\}} G'(z) \end{aligned}$$

and let $(U(x, t), V(x, t)) = (C_1 - u, C_2 - v)e^{-(a_{12}+M_2)t}$, then we have

$$\begin{aligned} G(u) &= G(C_1 - Ue^{(a_{12}+M_2)t}) \\ &= G(C_1) - G'(\xi(x, t))Ue^{(a_{12}+M_2)t}, \end{aligned}$$

where $\xi(x, t)$ is between C_1 and $u(x, t)$, and therefore (U, V) satisfies

$$\left\{ \begin{array}{ll} \frac{\partial U(x, t)}{\partial t} > d \frac{\partial^2 U(x, t)}{\partial x^2} - (a_{11} + a_{12} + M_2)U(x, t) \\ \quad + a_{12}V(x, t), & g(t) < x < h(t), \ 0 < t \leq T_0, \\ \frac{\partial V(x, t)}{\partial t} > -(a_{22} + a_{12} + M_2)V(x, t) \\ \quad + G'(\xi(x, t))U(x, t), & g(t) < x < h(t), \ 0 < t \leq T_0, \\ U(x, t) = C_1 e^{-(a_{12}+M_2)t}, & x = g(t) \text{ or } x = h(t), \ t > 0, \\ V(x, t) = C_2 e^{-(a_{12}+M_2)t}, & x = g(t) \text{ or } x = h(t), \ t > 0, \\ U(x, 0) \geq 0, \ V(x, 0) \geq 0, & -h_0 \leq x \leq h_0. \end{array} \right. \quad (2.3)$$

We now show that $\min\{U(x, t), V(x, t)\} \geq 0$ in $[g(t), h(t)] \times [0, T_0]$. Otherwise, there exists $(x_0, t_0) \in (g(t), h(t)) \times (0, T_0]$ such that

$$\min\{U(x_0, t_0), V(x_0, t_0)\} = \min_{[g(t), h(t)] \times [0, T_0]} \min\{U(x, t), V(x, t)\} < 0.$$

If $U(x_0, t_0) = \min\{U(x_0, t_0), V(x_0, t_0)\} < 0$, then $U(x, t)$ attains its minimum in $[g(t), h(t)] \times [0, T_0]$ at (x_0, t_0) ; therefore,

$$\frac{\partial U(x_0, t_0)}{\partial t} - d \frac{\partial^2 U(x_0, t_0)}{\partial x^2} \leq 0.$$

However, $-(a_{11} + a_{12} + M_2)U(x_0, t_0) + a_{12}V(x_0, t_0) \geq -(a_{11} + M_2)U(x_0, t_0) > 0$, which leads a contradiction to the first inequality in (2.3).

Similarly, if $V(x_0, t_0) = \min\{U(x_0, t_0), V(x_0, t_0)\} < 0$, then $V(x, t)$ attains its minimum in $[g(t), h(t)] \times [0, T_0]$ at (x_0, t_0) ; therefore, $\frac{\partial V(x_0, t_0)}{\partial t} \leq 0$. However, $-(a_{22} + a_{12} + M_2)V(x_0, t_0) + G'(\xi)U(x_0, t_0) \geq -(a_{22} + a_{12})V(x_0, t_0) > 0$, which leads to a contradiction to the second inequality in (2.3). Thus, we have $\min\{U(x, t), V(x, t)\} \geq 0$ in $[g(t), h(t)] \times [0, T_0]$, or $u(x, t) \leq C_1$ and $v(x, t) \leq C_2$ in $[g(t), h(t)] \times [0, T_0]$. \square

The next lemma shows that the left free boundary for (1.3) is strictly monotone decreasing and the right boundary is increasing.

Lemma 2.3 *Let $(u, v; g, h)$ be a solution to (1.3) defined for $t \in (0, T_0]$ for some $T_0 \in (0, +\infty)$. Then there exists a constant C_3 independent of T_0 such that*

$$0 < -g'(t), \quad h'(t) \leq C_3 \quad \text{for } t \in (0, T_0].$$

Proof: Applying the strong maximum principle to the equation of u gives

$$u_x(h(t), t) < 0 \quad \text{for } 0 < t \leq T_0.$$

Hence $h'(t) > 0$ for $t \in (0, T_0]$ by the free boundary condition in (1.3). Similarly, $g'(t) < 0$ for $t \in (0, T_0]$.

It remains to be shown that $-g'(t), h'(t) \leq C_3$ for $t \in (0, T_0]$ and some C_3 . The proof is similar to that of Lemma 2.2 in [10] with $C_3 = 2MC_1\mu$ and

$$M = \max \left\{ \frac{1}{h_0}, \sqrt{\frac{a_{12}C_2}{2dC_1}}, \frac{4\|u_0\|_{C^1([-h_0, h_0])}}{3C_1} \right\},$$

we omit it here. \square

Since u, v and $g'(t), h'(t)$ are bounded in $(g(t), h(t)) \times (0, T_0]$ by constants independent of T_0 , the global solution is guaranteed.

Theorem 2.4 *The solution of (1.3) exists and is unique for all $t \in (0, \infty)$.*

In what follows, we exhibit the comparison principle, which can be proved similarly to a Lemma 3.5 in [10].

Lemma 2.5 *(The Comparison Principle) Assume that $\bar{g}, \bar{h} \in C^1([0, +\infty))$, $\bar{u}(x, t), \bar{v}(x, t) \in C([\bar{g}(t), \bar{h}(t)] \times [0, +\infty)) \cap C^{2,1}((\bar{g}(t), \bar{h}(t)) \times (0, +\infty))$, and*

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} \geq d \frac{\partial^2 \bar{u}}{\partial x^2} - a_{11}\bar{u} + a_{12}\bar{v}, & \bar{g}(t) < x < \bar{h}(t), t > 0, \\ \frac{\partial \bar{v}}{\partial t} \geq -a_{22}\bar{v} + G(\bar{u}), & \bar{g}(t) < x < \bar{h}(t), t > 0, \\ \bar{u}(x, t) = \bar{v}(x, t) = 0, & x = \bar{g}(t) \text{ or } x = \bar{h}(t), t > 0, \\ \bar{g}(0) \leq -h_0, \bar{g}'(t) \leq -\mu \frac{\partial \bar{u}}{\partial x}(\bar{g}(t), t), & t > 0, \\ \bar{h}(0) \geq h_0, \bar{h}'(t) \geq -\mu \frac{\partial \bar{u}}{\partial x}(\bar{h}(t), t), & t > 0, \\ \bar{u}(x, 0) \geq u_0(x), \bar{v}(x, 0) \geq v_0(x), & -h_0 \leq x \leq h_0. \end{cases}$$

Then the solution $(u, v; g, h)$ to the free boundary problem (1.3) satisfies

$$\begin{aligned} h(t) &\leq \bar{h}(t), \quad g(t) \geq \bar{g}(t), \quad t \in [0, +\infty), \\ u(x, t) &\leq \bar{u}(x, t), \quad v(x, t) \leq \bar{v}(x, t), \quad (x, t) \in [g(t), h(t)] \times [0, +\infty). \end{aligned}$$

Remark 2.1 The pair (\bar{u}, \bar{h}) in Lemma 2.5 is usually called an upper solution of (1.3). We can define a lower solution by reversing all of the inequalities in the obvious places. Moreover, one can easily prove an analogue of Lemma 2.5 for lower solutions.

We next fix v_0, μ, a_{ij} , let $u_0 = \sigma\phi(x)$ and examine the dependence of the solution on σ , writing $(u^\sigma, v^\sigma; g^\sigma, h^\sigma)$ to emphasize this dependence. As a corollary of Lemma 2.5, we have the following monotonicity:

Corollary 2.6 Let $(u_0, v_0) = \sigma(\phi(x), \psi(x))$. For fixed $\phi(x), \psi(x), \mu$ and a_{ij} , if $\sigma_1 \leq \sigma_2$, then $u^{\sigma_1}(x, t) \leq u^{\sigma_2}(x, t)$ and $v^{\sigma_1}(x, t) \leq v^{\sigma_2}(x, t)$ in $[g^{\sigma_1}(t), h^{\sigma_1}(t)] \times (0, \infty)$, $g^{\sigma_1}(t) \geq g^{\sigma_2}(t)$ and $h^{\sigma_1}(t) \leq h^{\sigma_2}(t)$ in $(0, \infty)$.

3 Bacteria vanishing

It follows from Lemma 2.3 that $x = h(t)$ is monotonic increasing, $x = g(t)$ is monotonic decreasing and therefore there exist $h_\infty, -g_\infty \in (0, +\infty]$ such that $\lim_{t \rightarrow +\infty} h(t) = h_\infty$ and $\lim_{t \rightarrow +\infty} g(t) = g_\infty$. The next lemma shows that if $h_\infty < \infty$, then $-g_\infty < \infty$, and vice versa. That is, the double free boundary fronts $x = g(t)$ and $x = h(t)$ are both finite or infinite simultaneously.

Lemma 3.1 Let $(u, v; g, h)$ be a solution to (1.3) defined for $t \in [0, +\infty)$ and $x \in [g(t), h(t)]$. Then we have

$$-2h_0 < g(t) + h(t) < 2h_0 \text{ for } t \in [0, +\infty).$$

Proof: By continuity we know $g(t) + h(t) > -2h_0$ holds for small $t > 0$. Define

$$T := \sup\{s : g(t) + h(t) > -2h_0 \text{ for all } t \in [0, s)\}.$$

As in [12], we claim that $T = \infty$. Otherwise, $0 < T < \infty$ and

$$g(t) + h(t) > -2h_0 \text{ for } t \in [0, T), \quad g(T) + h(T) = -2h_0.$$

Hence,

$$g'(T) + h'(T) \leq 0. \tag{3.1}$$

To get a contradiction, we consider the functions

$$w(x, t) := u(x, t) - u(-x - 2h_0, t), \quad z(x, t) := v(x, t) - v(-x - 2h_0, t)$$

over the region

$$\Lambda := \{(x, t) : x \in [g(t), -h_0], t \in [0, T]\}.$$

It is easy to check that the pair (w, z) is well-defined for $(x, t) \in \Lambda$ since $-h_0 \leq -x - 2h_0 \leq -g(t) - 2h_0 \leq h(t)$, and the pair satisfies

$$w_t - dw_{xx} = -a_{11}w + a_{12}z \text{ for } g(t) < x < -h_0, \quad 0 < t \leq T,$$

$$z_t = c_{21}(x, t)w - a_{22}z \text{ for } g(t) < x < -h_0, \quad 0 < t \leq T$$

with $0 \leq c_{21} := \frac{G(u(x, t)) - G(u(-x - 2h_0, t))}{u(x, t) - u(-x - 2h_0, t)} \in L^\infty(\Lambda)$, and

$$w(-h_0, t) = z(-h_0, t) = 0, \quad w(g(t), t) < 0, \quad z(g(t), t) < 0 \text{ for } 0 < t < T.$$

Moreover,

$$w(g(T), T) = u(g(T), T) - u(-g(T) - 2h_0, T) = u(g(T), T) - u(h(T), T) = 0.$$

Applying the proof for the strong maximum principle and the Hopf lemma, we deduce

$$w(x, t) < 0, \quad z(x, t) < 0 \text{ in } (g(t), -h_0) \times (0, T] \text{ and } w_x(g(T), T) < 0.$$

However,

$$w_x(g(T), T) = \frac{\partial u}{\partial x}(g(T), T) + \frac{\partial u}{\partial x}(h(T), T) = -[g'(T) + h'(T)]/(\mu),$$

which implies

$$g'(T) + h'(T) > 0,$$

a contradiction to (3.1). Hence we have proven

$$g(t) + h(t) > -2h_0 \text{ for all } t > 0.$$

Analogously, we can prove $g(t) + h(t) < 2h_0$ for all $t > 0$ by considering

$$W(x, t) := u(x, t) - u(2h_0 - x, t), \quad Z(x, t) := v(x, t) - v(2h_0 - x, t)$$

over the region $\Lambda' := [h_0, h(t)] \times [0, T']$ with $T' := \sup\{s : g(t) + h(t) < 2h_0 \text{ for all } t \in [0, s]\}$. This completes the proof. \square

Next, we discuss the properties of the free boundary, because the transmission of the bacteria depends on whether $h_\infty - g_\infty = \infty$ and $\limsup_{t \rightarrow +\infty} (\|u(\cdot, t)\|_{C([g(t), h(t)])} + \|v(\cdot, t)\|_{C([g(t), h(t)])}) = 0$. We then have the following definitions:

Definition 3.1 *The bacteria are vanishing if*

$$h_\infty - g_\infty < \infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} (\|u(\cdot, t)\|_{C([g(t), h(t)])} + \|v(\cdot, t)\|_{C([g(t), h(t)])}) = 0,$$

and spreading if

$$h_\infty - g_\infty = \infty \quad \text{and} \quad \limsup_{t \rightarrow +\infty} (\|u(\cdot, t)\|_{C([g(t), h(t)])} + \|v(\cdot, t)\|_{C([g(t), h(t)])}) > 0.$$

The next result shows that if $h_\infty - g_\infty < \infty$, then vanishing occurs.

Lemma 3.2 *If $h_\infty - g_\infty < \infty$, then $\lim_{t \rightarrow +\infty} (\|u(\cdot, t)\|_{C([g(t), h(t)])} + \|v(\cdot, t)\|_{C([g(t), h(t)])}) = 0$.*

Proof: We first prove that $\lim_{t \rightarrow +\infty} \|u(\cdot, t)\|_{C([g(t), h(t)])} = 0$. Assume that

$$\limsup_{t \rightarrow +\infty} \|u(\cdot, t)\|_{C([g(t), h(t)])} = \delta > 0$$

by contradiction. Then there exists a sequence (x_k, t_k) in $(g(t), h(t)) \times (0, \infty)$ such that $u(x_k, t_k) \geq \delta/2$ for all $k \in \mathbb{N}$, and $t_k \rightarrow \infty$ as $k \rightarrow \infty$. Since $-\infty < g_\infty < g(t) < x_k < h(t) < h_\infty < \infty$, we then have that a subsequence of $\{x_n\}$ converges to $x_0 \in (g_\infty, h_\infty)$. Without loss of generality, we assume $x_k \rightarrow x_0$ as $k \rightarrow \infty$.

Define $W_k(x, t) = u(x, t_k + t)$ and $Z_k(x, t) = v(x, t_k + t)$ for $x \in (g(t_k + t), h(t_k + t))$, $t \in (-t_k, \infty)$. It follows from parabolic regularity that $\{(W_k, Z_k)\}$ has a subsequence $\{(W_{k_i}, Z_{k_i})\}$ such that $(W_{k_i}, Z_{k_i}) \rightarrow (\tilde{W}, \tilde{Z})$ as $i \rightarrow \infty$ and (\tilde{W}, \tilde{Z}) satisfies

$$\begin{cases} \tilde{W}_t - d\tilde{W}_{xx} = -a_{11}\tilde{W} + a_{12}\tilde{Z}, & g_\infty < x < h_\infty, \quad t \in (-\infty, \infty), \\ \tilde{Z}_t = -a_{22}\tilde{Z} + G(\tilde{W}), & g_\infty < x < h_\infty, \quad t \in (-\infty, \infty). \end{cases}$$

Note that $\tilde{W}(x_0, 0) \geq \delta/2$; therefore, $\tilde{W} > 0$ in $(g_\infty, h_\infty) \times (-\infty, \infty)$.

Using a similar method to prove the Hopf lemma at the point $(h_\infty, 0)$ yields $\tilde{W}_x(h_\infty, 0) \leq -\sigma_0$ for some $\sigma_0 > 0$.

On the other hand, since $-g(t)$ and $h(t)$ are increasing and bounded, it follows from standard L^p theory and then the Sobolev imbedding theorem ([16]) that for any $0 < \alpha < 1$, there exists a constant \tilde{C} depending on $\alpha, h_0, \|u_0\|_{C^2[-h_0, h_0]}, \|v_0\|_{C^2[-h_0, h_0]}$, and g_∞, h_∞ such that

$$\|u\|_{C^{1+\alpha, (1+\alpha)/2}([g(t), h(t)] \times [0, \infty))} + \|h\|_{C^{1+\alpha/2}([0, \infty))} \leq \tilde{C}. \quad (3.2)$$

Now, since $\|h\|_{C^{1+\alpha/2}([0, \infty))} \leq \tilde{C}$ and $h(t)$ is bounded, we then have $h'(t) \rightarrow 0$ as $t \rightarrow \infty$, that is, $\frac{\partial u}{\partial x}(h(t_k), t_k) \rightarrow 0$ as $t_k \rightarrow \infty$ by the free boundary condition.

Moreover, the fact that $\|u\|_{C^{1+\alpha, (1+\alpha)/2}([g(t), h(t)] \times [0, \infty))} \leq \tilde{C}$ gives $\frac{\partial u}{\partial x}(h(t_k), t_k + 0) = (W_k)_x(h(t_k), 0) \rightarrow \tilde{W}_x(h_\infty, 0)$ as $k \rightarrow \infty$, and then $\tilde{W}_x(h_\infty, 0) = 0$, which leads to a contradiction to the fact that $\tilde{W}_x(h_\infty, 0) \leq -\sigma_0 < 0$. Thus $\lim_{t \rightarrow +\infty} \|u(\cdot, t)\|_{C([g(t), h(t)])} = 0$.

Note that $v(x, t)$ satisfies

$$\frac{\partial v(x, t)}{\partial t} = -a_{22}v(x, t) + G(u(x, t)), \quad g(t) < x < h(t), \quad t > 0,$$

and $G(u(x, t)) \rightarrow 0$ uniformly for $x \in [g(t), h(t)]$ as $t \rightarrow \infty$; therefore, we have $\lim_{t \rightarrow +\infty} \|v(\cdot, t)\|_{C([g(t), h(t)])} = 0$. \square

In the introduction, a threshold R_0 , usually called the basic reproduction number, is given to decide whether the bacteria described by (1.2) vanish. Notice that the interval domain for free boundary problem (1.3) changes with t ; therefore, the basic reproduction number is not a constant and should change with t .

Now we introduce the basic reproduction number $R_0^F(t)$ for (1.3) by

$$R_0^F(t) := R_0^D((g(t), h(t))) = \frac{G'(0) \frac{a_{12}}{a_{22}}}{a_{11} + d(\frac{\pi}{h(t) - g(t)})^2},$$

where we use $R_0^D(\Omega)$ to denote the basic reproduction number for the corresponding problem in Ω with null Dirichlet boundary condition on $\partial\Omega$. Now, the following result is obvious; see also Lemma 2.3 in [14].

Lemma 3.3 *$1 - R_0^F(t)$ has the same sign as λ_0 , where λ_0 is the principal eigenvalue of the problem*

$$\begin{cases} -d\psi_{xx} = -a_{11}\psi + G'(0) \frac{a_{12}}{a_{22}}\psi + \lambda_0\psi, & x \in (g(t), h(t)), \\ \psi(x) = 0, & x = g(t) \text{ or } x = h(t). \end{cases} \quad (3.3)$$

In fact, here

$$\lambda_0 = a_{11} + d(\frac{\pi}{h(t) - g(t)})^2 - G'(0) \frac{a_{12}}{a_{22}} = [a_{11} + d(\frac{\pi}{h(t) - g(t)})^2](1 - R_0^F(t)).$$

With the above defined reproduction number, we also have

Lemma 3.4 *$R_0^F(t)$ is strictly monotone increasing function of t , that is if $t_1 < t_2$, then $R_0^F(t_1) < R_0^F(t_2)$. Moreover, if $h(t) \rightarrow \infty$ as $t \rightarrow \infty$, then $R_0^F(t) \rightarrow R_0$ as $t \rightarrow \infty$.*

Next we give sufficient conditions so that the bacteria are vanishing.

Theorem 3.5 *If $R_0 \leq 1$, then $h_\infty - g_\infty < \infty$ and $\lim_{t \rightarrow +\infty} (\|u(\cdot, t)\|_{C([g(t), h(t)])} + \|v(\cdot, t)\|_{C([g(t), h(t)])}) = 0$.*

Proof: We first show that $h_\infty - g_\infty < +\infty$. In fact, direct calculations yield

$$\begin{aligned}
& \frac{d}{dt} \int_{g(t)}^{h(t)} [u(x, t) + \frac{a_{12}}{a_{22}} v(x, t)] dx \\
&= \int_{g(t)}^{h(t)} [u_t + \frac{a_{12}}{a_{22}} v_t](x, t) dx + h'(t) [u + \frac{a_{12}}{a_{22}} v](h(t), t) - g'(t) [u + \frac{a_{12}}{a_{22}} v](g(t), t) \\
&= \int_{g(t)}^{h(t)} du_{xx} dx + \int_{g(t)}^{h(t)} -a_{11} u(x, t) + \frac{a_{12}}{a_{22}} G(u(x, t)) dx \\
&= -\frac{d}{\mu} (h'(t) - g'(t)) + \int_{g(t)}^{h(t)} -a_{11} u(x, t) + \frac{a_{12}}{a_{22}} G(u(x, t)) dx.
\end{aligned}$$

Integrating from 0 to t (> 0) gives

$$\begin{aligned}
\int_{g(t)}^{h(t)} [u + \frac{a_{12}}{a_{22}} v](x, t) dx &= \int_{g(0)}^{h(0)} [u + \frac{a_{12}}{a_{22}} v](x, 0) dx \\
&+ \frac{d}{\mu} (h(0) - g(0)) - \frac{d}{\mu} (h(t) - g(t))
\end{aligned} \tag{3.4}$$

$$+ \int_0^t \int_{g(s)}^{h(s)} -a_{11} u(x, t) + \frac{a_{12}}{a_{22}} G(u(x, t)) dx ds, \quad t \geq 0. \tag{3.5}$$

Since $\frac{G(z)}{z} \leq G'(0)$ by the assumption (A2), it follows from $R_0 \leq 1$ that $-a_{11} u(x, t) + \frac{a_{12}}{a_{22}} G(u(x, t)) \leq 0$ for $x \in [g(t), h(t)]$ and $t \geq 0$, we have

$$\frac{d}{\mu} (h(t) - g(t)) \leq \int_{g(0)}^{h(0)} [u + \frac{a_{12}}{a_{22}} v](x, 0) dx + \frac{d}{\mu} (h(0) - g(0))$$

for $t \geq 0$, which in turn gives that $h_\infty - g_\infty < \infty$. Therefore, the bacteria are vanishing as a consequence of Lemma 3.2. \square

Theorem 3.6 *If $R_0^F(0) < 1$ and $\|u_0(x)\|_{C([-h_0, h_0])}, \|v_0(x)\|_{C([-h_0, h_0])}$ are sufficiently small. Then $h_\infty - g_\infty < \infty$ and $\lim_{t \rightarrow +\infty} (\|u(\cdot, t)\|_{C([g(t), h(t)])} + \|v(\cdot, t)\|_{C([g(t), h(t)])}) = 0$.*

Proof: We construct a suitable upper solution for (u, v) . Since $R_0^F(0) < 1$, it follows from Lemma 3.3 that there is a $\lambda_0 > 0$ and $0 < \psi(x) \leq 1$ in $(-h_0, h_0)$ such that

$$\begin{cases} -d\psi_{xx} = -a_{11}\psi + G'(0)\frac{a_{12}}{a_{22}}\psi + \lambda_0\psi, & -h_0 < x < h_0, \\ \psi(x) = 0, & x = \pm h_0. \end{cases} \tag{3.6}$$

Therefore, there exists a small $\delta > 0$ such that

$$-\delta + \left(\frac{1}{(1+\delta)^2} - 1\right) - a_{11} + G'(0)\frac{a_{12}}{a_{22}} + \left[\frac{1}{(1+\delta)^2} - \frac{1}{4}\right]\lambda_0 \geq 0.$$

Similarly as in [10], we set

$$\sigma(t) = h_0(1 + \delta - \frac{\delta}{2}e^{-\delta t}), \quad t \geq 0,$$

and

$$\bar{u}(x, t) = \varepsilon e^{-\delta t} \psi(x h_0 / \sigma(t)), \quad -\sigma(t) \leq x \leq \sigma(t), \quad t \geq 0.$$

$$\bar{v}(x, t) = \left(\frac{G'(0)}{a_{22}} + \frac{\lambda_0}{4a_{12}}\right) \bar{u}(x, t), \quad -\sigma(t) \leq x \leq \sigma(t), \quad t \geq 0.$$

Direct computations yield

$$\begin{aligned} & \bar{u}_t - d \frac{\partial^2 \bar{u}}{\partial x^2} + a_{11} \bar{u} - a_{12} \bar{v} \\ &= -\delta \bar{u} - \varepsilon e^{-\delta t} \psi' \frac{x h_0 \sigma'(t)}{\sigma^2(t)} + \left(\frac{h_0}{\sigma(t)}\right)^2 \left[-a_{11} + G'(0) \frac{a_{12}}{a_{22}} + \lambda_0\right] \bar{u} \\ & \quad + \left[a_{11} - G'(0) \frac{a_{12}}{a_{22}} - \frac{\lambda_0}{4}\right] \bar{u} \\ & \geq \bar{u} \left\{ -\delta + \left(\frac{1}{(1+\delta)^2} - 1\right) - a_{11} + G'(0) \frac{a_{12}}{a_{22}} + \left[\frac{1}{(1+\delta)^2} - \frac{1}{4}\right] \lambda_0 \right\} \geq 0, \end{aligned}$$

$$\begin{aligned} & \bar{v}_t + a_{22} \bar{v} - G(\bar{u}(x, t)) \\ &= -\delta \bar{v} - \varepsilon e^{-\delta t} \psi' \frac{x h_0 \sigma'(t)}{\sigma^2(t)} \left(\frac{G'(0)}{a_{22}} + \frac{\lambda_0}{4a_{12}}\right) + a_{22} \left(\frac{G'(0)}{a_{22}} + \frac{\lambda_0}{4a_{12}}\right) \bar{u}(x, t) - G(\bar{u}(x, t)) \\ & \geq (a_{22} - \delta) \left(\frac{G'(0)}{a_{22}} + \frac{\lambda_0}{4a_{12}}\right) \bar{u}(x, t) - G(\bar{u}(x, t)) \\ &= (a_{22} - \delta) \left(\frac{G'(0)}{a_{22}} + \frac{\lambda_0}{4a_{12}}\right) \bar{u}(x, t) - G'(\xi(x, t)) \bar{u}(x, t) \\ &= [G'(0) - G'(\xi(x, t)) + (a_{22} - \delta) \frac{\lambda_0}{4a_{12}} - \frac{G'(0)}{a_{22}} \delta] \bar{u}(x, t) \end{aligned}$$

for all $t > 0$ and $-\sigma(t) < x < \sigma(t)$, where $\xi \in (0, \bar{u})$. Since $\bar{u} \leq \varepsilon$, if δ and ε are sufficiently small, then we have

$$[G'(0) - G'(\xi(x, t)) + (a_{22} - \delta) \frac{\lambda_0}{4a_{12}} - \frac{G'(0)}{a_{22}} \delta] \geq 0.$$

On the other hand, we have $\sigma'(t) = h_0 \frac{\delta^2}{2} e^{-\delta t}$, $-\bar{u}_x(\sigma(t), t) = -\varepsilon \frac{h_0}{\sigma(t)} \psi'(h_0) e^{-\delta t}$, and $-\bar{u}_x(-\sigma(t), t) = -\varepsilon \frac{h_0}{\sigma(t)} \psi'(-h_0) e^{-\delta t}$. Noticing that $\psi'(-h_0) = -\psi'(h_0)$, we now choose $\varepsilon = -\frac{\delta^2 h_0(1+\delta)}{2\mu\psi'(h_0)}$ such that

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} \geq d \frac{\partial^2 \bar{u}}{\partial x^2} - a_{11} \bar{u} + a_{12} \bar{v}, & -\sigma(t) < x < \sigma(t), t > 0, \\ \frac{\partial \bar{v}}{\partial t} \geq -a_{22} \bar{v} + G(\bar{u}), & -\sigma(t) < x < \sigma(t), t > 0, \\ \bar{u}(x, t) = \bar{v}(x, t) = 0, & x = \pm \sigma(t) t > 0, \\ -\sigma(0) < -h_0, -\sigma'(t) \leq -\mu \frac{\partial \bar{u}}{\partial x}(-\sigma(t), t), & t > 0, \\ \sigma(0) > h_0, \sigma'(t) \geq -\mu \frac{\partial \bar{u}}{\partial x}(\sigma(t), t), & t > 0. \end{cases}$$

If $\|u_0\|_{L^\infty} \leq \varepsilon \psi(\frac{h_0}{1+\delta/2})$ and $\|v_0\|_{L^\infty} \leq \varepsilon \psi(\frac{h_0}{1+\delta/2})(\frac{G'(0)}{a_{22}} + \frac{\lambda_0}{4a_{12}})$, then $u_0(x) \leq \varepsilon \psi(\frac{h_0}{1+\delta/2}) \leq \bar{u}(x, 0) = \varepsilon \psi(\frac{x}{1+\delta/2})$ and $v_0(x) \leq \varepsilon \psi(\frac{h_0}{1+\delta/2})(\frac{G'(0)}{a_{22}} + \frac{\lambda_0}{4a_{12}}) \leq \bar{v}(x, 0)$ for $x \in [-h_0, h_0]$. We can now apply Lemma 2.5 to conclude that $g(t) \geq -\sigma(t)$ and $h(t) \leq \sigma(t)$ for $t > 0$. It follows that $h_\infty - g_\infty \leq \lim_{t \rightarrow \infty} 2\sigma(t) = 2h_0(1+\delta) < \infty$, and $\lim_{t \rightarrow +\infty} (\|u(\cdot, t)\|_{C([g(t), h(t)])} + \|v(\cdot, t)\|_{C([g(t), h(t)])}) = 0$ by Lemma 3.2. \square

4 Bacteria spreading

In this section, we give the sufficient conditions for the bacteria to be spreading. We first prove that if $R_0^F(0) \geq 1$, the bacteria are spreading.

Theorem 4.1 *If $R_0^F(0) \geq 1$, then $h_\infty = -g_\infty = \infty$ and $\liminf_{t \rightarrow +\infty} \|u(\cdot, t)\|_{C([0, h(t)])} > 0$, that is, spreading occurs.*

Proof: We first consider the case that $R_0^F(0) := R_0^D((-h_0, h_0)) > 1$. In this case, we have that the eigenvalue problem

$$\begin{cases} -d\psi_{xx} = -a_{11}\psi + G'(0)\frac{a_{12}}{a_{22}}\psi + \lambda_0\psi, & -h_0 < x < h_0, \\ \psi(x) = 0, & x = \pm h_0 \end{cases} \quad (4.1)$$

admits a positive solution $\psi(x)$ with $\|\psi\|_{L^\infty} = 1$, where λ_0 is the principal eigenvalue. It follows from Lemma 3.3 that $\lambda_0 < 0$.

We construct a suitable lower solution to (1.3), and we define

$$\underline{u}(x, t) = \delta\psi(x), \quad \underline{v} = (\frac{G'(0)}{a_{22}} + \frac{\lambda_0}{4a_{12}})\delta\psi(x)$$

for $-h_0 \leq x \leq h_0$, $t \geq 0$, where δ is sufficiently small.

Direct computations yield

$$\begin{aligned}\frac{\partial \underline{u}}{\partial t} - d \frac{\partial^2 \underline{u}}{\partial x^2} + a_{11} \underline{u} - a_{12} \underline{v} &= \delta \psi(x) \left(\frac{3}{4} \lambda_0 \right) \leq 0 \\ \frac{\partial \underline{v}}{\partial t} + a_{22} \underline{v} - G(\underline{u}) &= \delta \psi(x) [G'(0) - G'(\xi(x, t)) + \frac{a_{22} \lambda_0}{4a_{12}}]\end{aligned}$$

for all $t > 0$ and $-h_0 < x < h_0$, where $\xi \in (0, \underline{u})$. Noting that $\lambda_0 < 0$ and $0 \leq \xi(x, t) \leq \underline{u}(x, t) \leq \delta$, we can chose δ sufficiently small such that

$$\left\{ \begin{array}{ll} \frac{\partial \underline{u}}{\partial t} \leq d \frac{\partial^2 \underline{u}}{\partial x^2} - a_{11} \underline{u} + a_{12} \underline{v}, & -h_0 < x < h_0, t > 0, \\ \frac{\partial \underline{v}}{\partial t} \leq -a_{22} \underline{v} + G(\underline{u}), & -h_0 < x < h_0, t > 0, \\ \underline{u}(x, t) = \underline{v}(x, t) = 0, & x = \pm h_0, t > 0, \\ 0 = -h'_0 \geq -\mu \frac{\partial \underline{u}}{\partial x}(-h_0, t), & t > 0, \\ 0 = h'_0 \leq -\mu \frac{\partial \underline{u}}{\partial x}(h_0, t), & t > 0, \\ \underline{u}(x, 0) \leq u_0(x), \underline{v}(x, 0) \leq v_0(x), & -h_0 \leq x \leq h_0. \end{array} \right.$$

Hence, applying Remark 2.1 yields that $u(x, t) \geq \underline{u}(x, t)$ and $v(x, t) \geq \underline{v}(x, t)$ in $[-h_0, h_0] \times [0, \infty)$. It follows that $\liminf_{t \rightarrow +\infty} \|u(\cdot, t)\|_{C([g(t), h(t)])} \geq \delta \psi(0) > 0$ and therefore $h_\infty - g_\infty = +\infty$ by Lemma 3.2.

If $R_0^F(0) = 1$, then for any positive time t_0 , we have $g(t_0) < -h_0$ and $h(t_0) > h_0$; therefore, $R_0^F(t_0) > R_0^F(0) = 1$ by the monotonicity in Lemma 3.4. Replacing the initial time 0 by the positive time t_0 , we then have $h_\infty - g_\infty = +\infty$ as above. \square

Remark 4.1 *It follows from the above proof that spreading occurs, if there exists $t_0 \geq 0$ such that $R_0^F(t_0) \geq 1$.*

Theorem 3.6 shows if $R_0^F(0) < 1$, vanishing occurs for small initial size of infected bacteria, and Theorem 3.5 implies that if $R_0 \leq 1$, vanishing always occurs for any initial values. The next result shows that spreading occurs for large values.

Theorem 4.2 *Suppose that $R_0^F(0) < 1 < R_0$. Then $h_\infty - g_\infty = \infty$ and $\liminf_{t \rightarrow +\infty} \|u(\cdot, t)\|_{C([0, h(t)])} > 0$ if $\|u_0(x)\|_{C([-h_0, h_0])}$ and $\|v_0(x)\|_{C([-h_0, h_0])}$ are sufficiently large.*

Proof: We construct a vector $(\underline{u}, \underline{v}, \underline{h})$ such that $u \geq \underline{u}$, $v \geq \underline{v}$ in $[-\underline{h}(t), \underline{h}(t)] \times [0, T_0]$, and also $g(t) \leq -\underline{h}(t)$, $h(t) \geq \underline{h}(t)$ in $[0, T_0]$. If we can choose T_0 such that $R_0^D((-\underline{h}(\sqrt{T_0}), \underline{h}(\sqrt{T_0}))) > 1$, then $h_\infty - g_\infty = \infty$.

We first consider the following eigenvalue problem

$$\begin{cases} -d\psi'' - \frac{1}{2}\psi' = \mu_0\psi, & 0 < x < 1, \\ \psi(0) = \psi(1) = 0. \end{cases} \quad (4.2)$$

It is well known that the principal eigenvalue μ_0 of this problem is simple; the corresponding eigenfunction $\psi(x)$ can be chosen to be positive in $[0, 1)$ and $\|\psi\|_{L^\infty} = 1$. It is also easy to see that $\mu_0 > \frac{1}{16d}$ and $\psi'(x) < 0$ in $(0, 1]$. Extending ψ in $[0, 1]$ to an even function ϕ in $[-1, 1]$ yields

$$\begin{cases} -d\phi'' - \frac{\text{sgn}(x)}{2}\phi' = \mu_0\phi, & -1 < x < 1, \\ \phi(-1) = \phi(1) = 0. \end{cases} \quad (4.3)$$

We now construct a suitable lower solution to (1.3) and we define

$$\underline{h}(t) = \sqrt{t + \delta}, \quad t \geq 0,$$

$$\underline{u}(x, t) = \frac{M}{(t + \delta)^k} \phi\left(\frac{x}{\sqrt{t + \delta}}\right), \quad -\sqrt{t + \delta} \leq x \leq \sqrt{t + \delta}, \quad t \geq 0,$$

$$\underline{v}(x, t) = \frac{M}{(t + \delta)^k} \phi\left(\frac{x}{\sqrt{t + \delta}}\right), \quad -\sqrt{t + \delta} \leq x \leq \sqrt{t + \delta}, \quad t \geq 0,$$

where δ, M, T_0, k are chosen as follows :

$$0 < \delta \leq \min\{1, h_0^2\}, \quad R_0^D((-\sqrt{T_0}, \sqrt{T_0})) > 1,$$

$$k \geq \max\{\mu_0 + a_{11}(T_0 + 1), a_{22}(T_0 + 1)\}, \quad -2\mu M\phi'(1) > (T_0 + 1)^k.$$

Here we have used the assumption that $R_0 > 1$ and the fact that $R_0^D((-\sqrt{T_0}, \sqrt{T_0})) \rightarrow R_0$ as $T_0 \rightarrow \infty$ by Lemma 3.4.

Direct computations yield

$$\begin{aligned} & \frac{\partial \underline{u}}{\partial t} - d \frac{\partial^2 \underline{u}}{\partial x^2} + a_{11}\underline{u} - a_{12}\underline{v} \\ &= -\frac{M}{(t + \delta)^{k+1}} \left[d\phi'' + \frac{x}{2\sqrt{t + \delta}}\phi' + (k + (-a_{11} + a_{12})(t + \delta))\phi \right] \\ &\leq -\frac{M}{(t + \delta)^{k+1}} \left[d\phi'' + \frac{\text{sgn}(x)}{2}\phi' + \mu_0\phi \right] \\ &= 0, \end{aligned}$$

$$\begin{aligned} & \frac{\partial \underline{v}}{\partial t} + a_{22}\underline{v} - G(\underline{u}(x, t)) \\ &\leq -\frac{M}{(t + \delta)^{k+1}} [(k + (-a_{22} + G(\underline{u})/\underline{u})(t + \delta))\phi] \\ &\leq -\frac{M}{(t + \delta)^{k+1}} [k - a_{22}(T_0 + 1)] \\ &\leq 0 \end{aligned}$$

for all $0 < t \leq T_0$ and $-\underline{h} < x < \underline{h}$.

$$\underline{h}'(t) + \mu \underline{u}_x(\sqrt{t+\delta}, t) = \frac{1}{2\sqrt{t+\delta}} + \frac{\mu M}{(t+\delta)^{k+1/2}} \phi'(1) < 0.$$

Then we have

$$\left\{ \begin{array}{ll} \frac{\partial \underline{u}}{\partial t} \leq d \frac{\partial^2 \underline{u}}{\partial x^2} - a_{11} \underline{u} + a_{12} \underline{v}, & -\underline{h} < x < \underline{h}, 0 < t \leq T_0, \\ \frac{\partial \underline{v}}{\partial t} \leq -a_{22} \underline{v} + G(\underline{u}(x, t)), & -\underline{h} < x < \underline{h}, 0 < t \leq T_0, \\ \underline{u}(x, t) = \underline{v}(x, t) = 0, & x = \pm \underline{h}(t), 0 < t \leq T_0, \\ -\underline{h}_0 = -\sqrt{\delta} \geq -h_0, & 0 < t \leq T_0, \\ \underline{h}_0 = \sqrt{\delta} \leq h_0, & 0 < t \leq T_0, \\ \underline{h}'(t) < -\mu \underline{u}_x(\sqrt{t+\delta}, t), & 0 < t \leq T_0, \\ -\underline{h}'(t) > -\mu \underline{u}_x(-\sqrt{t+\delta}, t), & 0 < t \leq T_0. \end{array} \right.$$

If $\underline{u}(x, 0) = \frac{M}{\delta^k} \phi(\frac{x}{\sqrt{\delta}}) < u_0(x)$ and $\underline{v}(x, 0) = \frac{M}{\delta^k} \phi(\frac{x}{\sqrt{\delta}}) < v_0(x)$ in $[0, \sqrt{\delta}]$, then using Lemma 2.5 yields $h(t) \geq \underline{h}(t)$ and $g(t) \leq -\underline{h}(t)$ in $[0, T_0]$. In particular, $h(T_0) - g(T_0) \geq 2\underline{h}(T_0) = 2\sqrt{T_0 + \delta} \geq 2\sqrt{T_0}$. Noting that

$$\begin{aligned} R_0^F(T_0) &:= R_0^D((g(T_0), h(T_0))) \geq R_0^D((-\underline{h}(T_0), \underline{h}(T_0))) \\ &\geq R_0^D((-\sqrt{T_0}, \sqrt{T_0})) > 1, \end{aligned}$$

we then have $h_\infty - g_\infty = +\infty$ by Theorem 4.1. \square

Theorem 4.3 (*Sharp threshold*) Suppose that $R_0 > 1$, with fixed μ , h_0 and (ϕ, ψ) satisfying (1.4). Let $(u, v; g, h)$ be a solution of (1.3) with $(u_0, v_0) = (\sigma\phi(x), \sigma\psi(x))$ for some $\sigma > 0$. Then there exists $\sigma^* = \sigma^*(\phi, \psi) \in [0, \infty)$ such that spreading occurs when $\sigma > \sigma^*$, and vanishing occurs when $0 < \sigma \leq \sigma^*$.

Proof: It follows from Theorem 4.1 that spreading always occurs if $R_0^F(0) \geq 1$. Hence, in this case we have $\sigma^*(\phi, \psi) = 0$ for any ϕ and ψ .

For the remaining case $R_0^F(0) < 1$, define

$$\sigma^* := \sup\{\sigma_0 : h_\infty(\sigma\phi, \sigma\psi) < \infty \text{ for } \sigma \in (0, \sigma_0]\}.$$

By Theorem 3.6, we see that in this case vanishing occurs for all small $\sigma > 0$; therefore, $\sigma^* \in (0, \infty]$. On the other hand, it follows from Theorem 4.2 that in this case spreading occurs for all large σ . Therefore, $\sigma^* \in (0, \infty)$, spreading occurs when $\sigma > \sigma^*$, and vanishing occurs when $0 < \sigma < \sigma^*$ by Corollary 2.6.

We claim that vanishing occurs when $\sigma = \sigma^*$. Otherwise $h_\infty - g_\infty = \infty$ for $\sigma = \sigma^*$. Since $R_0^F(t) \rightarrow R_0 > 1$ as $t \rightarrow \infty$, there exists $T_0 > 0$ such

that $R_0^F(T_0) > 1$. By the continuous dependence of $(u, v; g, h)$ on its initial values, we can find $\epsilon > 0$ sufficiently small so that the solution of (1.3) with $(u_0, v_0) = (\sigma^* - \epsilon)(\phi(x), \psi(x))$, denoted by $(u_\epsilon, v_\epsilon; g_\epsilon, h_\epsilon)$ satisfies $R_0^F(T_0) > 1$. This implies that spreading occurs for $(u_\epsilon, v_\epsilon; g_\epsilon, h_\epsilon)$, contradicting the definition of σ^* . This completes the proof. \square

Similarly, if we consider μ instead of u_0 as a varying parameter, the following result holds; see also Theorem 4.4 in [11].

Theorem 4.4 (*Sharp threshold*) Suppose that $R_0 > 1$, with fixed h_0 , u_0 and v_0 . Then there exists $\mu^* \in [0, \infty)$ such that spreading occurs when $\mu > \mu^*$, and vanishing occurs when $0 < \mu \leq \mu^*$.

Next, we consider the asymptotic behavior of the solution to (1.3) when the spreading occurs.

Theorem 4.5 Suppose that $R_0 > 1$. If spreading occurs, then the solution of free boundary problem (1.3) satisfies $\lim_{t \rightarrow +\infty} (u(x, t), v(x, t)) = (u^*, v^*)$ uniformly in any bounded subset of $(-\infty, \infty)$, where (u^*, v^*) is the unique positive equilibrium of (1.2).

Proof: (1) The limit superior of the solution

We recall that the comparison principle gives $(u(x, t), v(x, t)) \leq (\bar{u}(t), \bar{v}(t))$ for $(x, t) \in [g(t), h(t)] \times (0, \infty)$, where $(\bar{u}(t), \bar{v}(t))$ is the solution of the problem

$$\begin{cases} \bar{u}'(t) = -a_{11}\bar{u}(t) + a_{12}\bar{v}(t), & t > 0, \\ \bar{v}'(t) = -a_{22}\bar{v}(t) + G(\bar{u}(t)), & t > 0, \\ \bar{u}(0) = \|u_0\|_{L^\infty[-h_0, h_0]}, \quad \bar{v}(0) = \|v_0\|_{L^\infty[-h_0, h_0]}. \end{cases} \quad (4.4)$$

Since $R_0 > 1$, the unique positive equilibrium (u^*, v^*) is globally stable for the ODE system (4.4) and $\lim_{t \rightarrow \infty} (\bar{u}(t), \bar{v}(t)) = (u^*, v^*)$; therefore we deduce

$$\limsup_{t \rightarrow +\infty} (u(x, t), v(x, t)) \leq (u^*, v^*) \quad (4.5)$$

uniformly for $x \in (-\infty, \infty)$.

(2) The lower bound of the solution for a large time.

Note that $R_0 > 1$ and

$$\lim_{l \rightarrow \infty} \frac{G'(0) \frac{a_{12}}{a_{22}}}{a_{11} + d(\frac{\pi}{2l})^2} = R_0 > 1;$$

therefore, there is L_0 such that $\frac{G'(0) \frac{a_{12}}{a_{22}}}{a_{11} + d(\frac{\pi}{2L_0})^2} > 1$. This implies that the principal eigenvalue λ_0^* of

$$\begin{cases} -d\psi_{xx} = -a_{11}\psi + G'(0) \frac{a_{12}}{a_{22}}\psi + \lambda_0^*\psi, & x \in (-L_0, L_0), \\ \psi(x) = 0, & x = \pm L_0 \end{cases} \quad (4.6)$$

satisfies

$$\lambda_0^* = a_{11} + d\left(\frac{\pi}{2L_0}\right)^2 - G'(0)\frac{a_{12}}{a_{22}} < 0.$$

Since $h_\infty - g_\infty = \infty$ by assumption, $h_\infty = g_\infty = \infty$ by Lemma 3.1. Thus, for any $L \geq L_0$, there exists $t_L > 0$ such that $g(t) \leq -L$ and $h(t) \geq L$ for $t \geq t_L$.

Letting $\underline{U} = \delta\psi$ and $\underline{V} = G(\underline{U})/a_{22}$, we can choose δ sufficiently small such that $(\underline{U}, \underline{V})$ satisfies

$$\begin{cases} \underline{U}_t \leq d\underline{U}_{xx} - a_{11}\underline{U} + a_{12}\underline{V}, & -L_0 < x < L_0, \ t > t_{L_0}, \\ \underline{V}_t = -a_{22}\underline{V} + G(\underline{U}), & -L_0 < x < L_0, \ t > t_{L_0}, \\ \underline{U}(x, t) = 0, & x = \pm L_0, \ t > t_{L_0}, \\ \underline{U}(x, t_{L_0}) \leq u(x, t_{L_0}), \ \underline{V}(x, t_{L_0}) = v(x, t_{L_0}), & -L_0 \leq x \leq L_0, \end{cases}$$

meaning that $(\underline{U}, \underline{V})$ is a lower solution of the solution (u, v) in $[-L_0, L_0] \times [t_{L_0}, \infty)$. We then have $(u, v) \geq (\delta\psi, G(\delta\psi)/a_{22})$ in $[-L_0, L_0] \times [t_{L_0}, \infty)$, which implies that the solution can not decay to zero.

(3) The limit inferior of the solution.

We extend $\psi(x)$ to $\psi_{L_0}(x)$ by defining $\psi_{L_0}(x) := \psi(x)$ for $-L_0 \leq x \leq L_0$ and $\psi_{L_0}(x) := 0$ for $x < -L_0$ or $x > L_0$. Now for $L \geq L_0$, (u, v) satisfies

$$\begin{cases} u_t = du_{xx} - a_{11}u + a_{12}v, & g(t) < x < h(t), \ t > t_L, \\ v_t = -a_{22}v + G(u), & g(t) < x < h(t), \ t > t_L, \\ u(x, t) = 0, & x = g(t) \text{ or } x = h(t), \ t > t_L, \\ u(x, t_L) \geq \delta\psi_{L_0}, \ v(x, t_L) \geq G(\delta\psi_{L_0})/a_{22}, & -L \leq x \leq L; \end{cases} \quad (4.7)$$

therefore, we have $(u, v) \geq (\underline{u}, \underline{v})$ in $[-L, L] \times [t_L, \infty)$, where $(\underline{u}, \underline{v})$ satisfies

$$\begin{cases} \underline{u}_t = d\underline{u}_{xx} - a_{11}\underline{u} + a_{12}\underline{v}, & -L < x < L, \ t > t_L, \\ \underline{v}_t = -a_{22}\underline{v} + G(\underline{u}), & -L < x < L, \ t > t_L, \\ \underline{u}(x, t) = 0, & x = \pm L, \ t > t_L, \\ \underline{u}(x, t_L) = \delta\psi_{L_0}, \ \underline{v}(x, t_L) = G(\delta\psi_{L_0})/a_{22}, & -L \leq x \leq L. \end{cases} \quad (4.8)$$

The system (4.8) is quasimonotone increasing; therefore, it follows from the upper and lower solution method and the theory of monotone dynamical systems ([20] Corollary 3.6) that $\lim_{t \rightarrow +\infty} (\underline{u}(x, t), \underline{v}(x, t)) \geq (\underline{u}_L(x), \underline{v}_L(x))$ uniformly in $[-L, L]$, where $(\underline{u}_L, \underline{v}_L)$ satisfies

$$\begin{cases} -d\underline{u}_L'' = -a_{11}\underline{u}_L + a_{12}\underline{v}_L, & -L < x < L, \\ -a_{22}\underline{v}_L + G(\underline{u}_L) = 0, & -L < x < L, \\ \underline{u}_L(x) = 0, & x = \pm L \end{cases} \quad (4.9)$$

and is the minimum upper solution $(\delta\psi_{L_0}, G(\delta\psi_{L_0})/a_{22})$.

Now we give the monotonicity and show that if $0 < L_1 < L_2$, then $\underline{u}_{L_1}(x) \leq \underline{u}_{L_2}(x)$ in $[-L_1, L_1]$. The result is derived by comparing the boundary conditions and initial conditions in (4.8) for $L = L_1$ and $L = L_2$.

Let $L \rightarrow \infty$. By classical elliptic regularity theory and a diagonal procedure, it follows that $(\underline{u}_L(x), \underline{v}_L(x))$ converges uniformly on any compact subset of $(-\infty, \infty)$ to $(\underline{u}_\infty, \underline{v}_\infty)$ that is continuous on $(-\infty, \infty)$ and satisfies

$$\begin{cases} -d\underline{u}_\infty'' = -a_{11}\underline{u}_\infty + a_{12}\underline{v}_\infty, & -\infty < x < \infty, \\ -a_{22}\underline{v}_\infty + G(\underline{u}_\infty) = 0, & -\infty < x < \infty, \\ \underline{u}_\infty(x) \geq \delta\psi_{L_0}, \quad \underline{v}_\infty(x) \geq G(\delta\psi_{L_0})/a_{22}, & -\infty < x < \infty. \end{cases}$$

Next, we observe that $\underline{u}_\infty(x) \equiv u^*$ and $\underline{v}_\infty(x) \equiv v^*$, which can be derived by considering the problem

$$-dw'' = -a_{11}w + \frac{a_{12}}{a_{22}}G(w).$$

The uniqueness of the positive solution follows from the assumption on G and the condition $R_0 > 1$.

Now for any given $[-M, M]$ with $M \geq L_0$, since that $(\underline{u}_L(x), \underline{v}_L(x)) \rightarrow (u^*, v^*)$ uniformly in $[-M, M]$, which is the compact subset of $(-\infty, \infty)$, as $L \rightarrow \infty$, we deduce that for any $\varepsilon > 0$, there exists $L^* > L_0$ such that $(\underline{u}_{L^*}(x), \underline{v}_{L^*}(x)) \geq (u^* - \varepsilon, v^* - \varepsilon)$ in $[-M, M]$. As above, there is t_{L^*} such that $[g(t), h(t)] \supseteq [-L^*, L^*]$ for $t \geq t_{L^*}$. Therefore,

$$(u(x, t), v(x, t)) \geq (\underline{u}(x, t), \underline{v}(x, t)) \text{ in } [-L^*, L^*] \times [t_{L^*}, \infty),$$

and

$$\lim_{t \rightarrow +\infty} (\underline{u}(x, t), \underline{v}(x, t)) \geq (\underline{u}_{L^*}(x), \underline{v}_{L^*}(x)) \text{ in } [-L^*, L^*].$$

Using the fact that $(\underline{u}_{L^*}(x), \underline{v}_{L^*}(x)) \geq (u^* - \varepsilon, v^* - \varepsilon)$ in $[-M, M]$ gives

$$\liminf_{t \rightarrow +\infty} (u(x, t), v(x, t)) \geq (u^* - \varepsilon, v^* - \varepsilon) \text{ in } [-M, M].$$

Since $\varepsilon > 0$ is arbitrary, we then have $\liminf_{t \rightarrow +\infty} u(x, t) \geq u^*$ and $\liminf_{t \rightarrow +\infty} v(x, t) \geq v^*$ uniformly in $[-M, M]$, which together with (4.5) imply that $\lim_{t \rightarrow +\infty} u(x, t) = u^*$ and $\lim_{t \rightarrow +\infty} v(x, t) = v^*$ uniformly in any bounded subset of $(-\infty, \infty)$. \square

Combining Remark 4.1, Theorem 4.2 and Theorem 4.5, we immediately obtain the following spreading-vanishing dichotomy:

Theorem 4.6 *Suppose that $R_0 > 1$. Let $(u(x, t), v(x, t); g(t), h(t))$ be the solution of free boundary problem (1.3). Then, the following alternatives hold:*

Either

(i) Spreading: $h_\infty - g_\infty = +\infty$ and $\lim_{t \rightarrow +\infty} (u(x, t), v(x, t)) = (u^*, v^*)$ uniformly in any bounded subset of $(-\infty, \infty)$;

or

(ii) Vanishing: $h_\infty - g_\infty \leq h^*$ with $\frac{G'(0) \frac{a_{12}}{a_{22}}}{a_{11} + d(\frac{\pi}{h^*})^2} = 1$ and $\lim_{t \rightarrow +\infty} (\|u(\cdot, t)\|_{C([g(t), h(t)])} + \|v(\cdot, t)\|_{C([g(t), h(t)])}) = 0$.

5 Discussion

In this paper, a free boundary problem is used to describe the expanding of bacteria in a man-environment-man epidemic model in a one-dimensional habitat. We take into account the spreading and vanishing of the bacteria. Here, vanishing implies not only that the bacteria disappear eventually, but also that the infected habitat is limited, and spreading means the existence of the bacteria in the long run with an uncontrollable infected environment. Sufficient conditions for the bacteria spreading or vanishing are given.

Compared with existing work described by reaction-diffusion systems (1.1) in [4] or established by travelling waves and entire solutions in [21, 23, 24, 25, 26, 27], our model (1.3) provides a different way to understand the expanding process of bacteria. It is well-known that for the ODE system (1.2), the basic reproduction number $R_0 := \frac{G'(0)a_{12}}{a_{11}a_{22}}$ determines whether the bacteria die out ($R_0 < 1$) or remain endemic ($R_0 > 1$). However, in our problem (1.3), the infected habitat is changing with time t ; therefore, we introduced the basic reproduction number

$$R_0^F(t) := \frac{G'(0) \frac{a_{12}}{a_{22}}}{a_{11} + d(\frac{\pi}{h(t)-g(t)})^2},$$

which depends on the habitat $(g(t), h(t))$, the diffusion rate d and the coefficients in (1.3). We showed that $R_0^F(t) \leq R_0$ and $R_0^F(t) \rightarrow R_0$ if $(g(t), h(t)) \rightarrow (-\infty, +\infty)$ as $t \rightarrow \infty$. Furthermore, if $R_0 \leq 1$, the bacteria are always vanishing (Theorem 3.5). The result is the same as that for the corresponding ODE system (1.2). However, if $R_0^F(t_0) \geq 1$ for some $t_0 \geq 0$, the bacteria are spreading (Theorem 4.1 and Remark 4.1). For the case $R_0^F(0) < 1 < R_0$, the spreading or vanishing of the bacteria depends on the initial size of bacteria (Theorem 4.3), or the ratio (μ) of the expansion speed of the free boundary and the population gradient at the expanding fronts (Theorem 4.4).

Ecologically, our main results reveal that if the multiplicative factor of the infectious bacteria is small, the bacteria will die out eventually and the humans are safe. Otherwise, the spreading or vanishing of the bacteria depends on the initial infected habitat, the diffusion rate, and other factors. In particular, the

initial number of bacteria plays a key role. A large initial number can induce the spreading of bacteria easily. A similar result obtained for an invasive species has been supported by substantial empirical evidence; see [10]. Therefore, we hope our model and theoretical results can be used to provide better prediction and prevention of infecting bacteria.

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